Variations of the secretary problem via Game Theory and Linear Programming

MELISSA DE CARVALHO[1]
LUCAS MONTEIRO CHAVES[1]
RICARDO MARTINS DE ABREU SILVA[2]

UFLA - Federal University of Lavras
DEX - Department of Exact Sciences
DCC - Department of Computer Science
P.O. Box 3037 - Campus da UFLA 37200-000 - Lavras (MG)- Brazil
[1]lucas@ufla.br
[2]ricardo.mabreu@gmail.com

Abstract. This paper presents models for three variants of the secretary problem based on a strategic form of zero-sum finite games for two players. Based on the minimax theorem for finite games, the problem of maximizing the minimum average payoff of a player, in spite of the strategies of the other player, is represented by a linear programming model, which solution using the simplex method presents not only one optimum strategy to the player, but validates some strategies also as optimal.

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1 Introduction

The secretary problem has become widely known after addressed by Martin Gardner, on his column of Mathematical Games at Scientific American Journal, in an issue of 1960 (Ferguson [3]).

The problem is: n candidates are interested in occupying a single secretary position. The candidates are interviewed, in a random order, however, just after each interview, it is decided to accept or reject the candidate. If one decides to reject a candidate, he/she cannot be accepted later and, once accepted, all others are rejected. Finally, if the first n-1 candidates are rejected, the n-th candidate is automatically accepted. Stated the problem, there is the need to answer the following question: What strategy should be adopted aiming to maximize the probability of hiring the best candidate? Or, at least, a sufficiently qualified one?

There are many variations of the secretary problem. Among them It can be distinguished the cases where the observer (who wants to hire the secretary) is playing against an opponent. In these cases, the observer wants to hire the best secretary (lower rank) or, at least, a sufficiently qualified (low rank). The role of the opponent is to choose the presentation order of the candidates, in order to maximize the rank of candidate accepted by the observer. Thus, it minimizes the likelihood of success of the observer.

In this work three variations of the secretary problem with one opponent are considered:

1. **First variation** - The opponent can choose any line at random of the $n \times n$ cyclical Latin square which, in turn, represents the order of the ranks of the candidates presented for the observer;
2. **Second variation** - The opponent chooses only the position of the best candidate;
3. **Third variation** - The opponent presents, always with probability $1/2$, the best or the worst secretary among those who have not yet been interviewed.

This paper presents models for the three variations of the secretary problem described above (Section 3) from the strategic point of view for zero-sum finite games for two players (Section 2).
There are no new results in this article. What is new is a surprisingly elementary approach that allows an intuitive comprehension to the problem. Such approach is strong enough to perform all proofs, even the less trivial ones. Therefore, one doesn't need great familiarity with stochastic optimization to understand this article. This is the reason for so short reference bibliography. Much more complete bibliography can be found in Freeman [5].
Based on the minimax theorem for finite games, the problem of maximizing the lowest average gain of the observer, no matter what is the strategy adopted by the opponent, is represented by a linear programming model (Section 4), whose solution via simplex method presents not only a good optimal strategy for the observer, but can also validate an optimum strategy of the observer (Section 5). Finally, the conclusions are outlined in Section 6. This article is based on the works of Ferguson [4] and Carvalho [2]. Related results can be found in Brighenti [1].

2 Strategic Model for Zero-Sum Finite Games for Two Players

The strategic form of the game is defined by three components (Ferguson [4]):

- The set of players $N = \{1, 2, 3, ..., n\}$;
- A sequence $A_1, ..., A_n$ of sets of strategies for the players;
- Assuming that the player 1 chooses the strategy $a_1 \in A_1$, the player 2 chooses the strategy $a_2 \in A_2$ and so on until $a_n \in A_n$ is chosen by the player n, the gain function (*payoff*) of the j-th player ($j = 1, ..., n$) is denoted by $f_j(a_1, ..., a_n)$. Therefore, the sequence $f_1(a_1, ..., a_n), ..., f_n(a_1, ..., a_n)$ of *payoff* functions for the players is the third component of the strategic form of a game.

A game in the strategic form is called zero-sum if the sum of earnings (*payoff*) of the players is always zero, despite of the actions taken by players. That is, the game is called zero-sum if and only if

$$\sum_{i=1}^{n} f_i(a_1, a_2, ..., a_n) = 0$$

for all $a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n$.

For zero-sum games with two players, the amount one wins is exactly what the other looses. The strategic form can be simplified to a triple (X, Y, Z), where:

- X is the non-empty set of strategies for the player I;
- Y is the non-empty set of strategies for the player II;
- $A : X \times Y \rightarrow \Re$ is the gain function *payoff* of the player I over the player II. Therefore, after the simultaneous choice of the strategy $x \in X$ by the player I and $y \in Y$ by the player II, $A(x, y)$ is the amount won by the player I payed by the player II. If $A(x, y)$ is negative, player I must pay the absolute value of this amount to player II.

If the sets X, Y are finite, we have **finite games**. In the particular case of sum-zero finite games, they are also called **matrix games**, because the *payoff* function can be represented by a matrix. That is, if $X = \{x_1, \ldots, x_m\}$, and $Y = \{y_1, \ldots, y_n\}$, then the game matrix or *payoff* matrix $A_{m\times n}$ can be represented as follows:

$$A = \begin{array}{cc} \begin{array}{ccc} y_1 & \cdots & y_n \end{array} & \\ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} x_1 \\ \vdots \\ x_m \end{array} \end{array}$$

where $a_{ij} = A(x_i, y_i)$.
If player I chooses a line and player II chooses a column, player II pays player I the correspondent matrix entry.
The elements of the sets X and Y are considered **pure strategies**. A **mixed strategie** is to choose at random a pure strategy to be used at each stage of the game.
For example, consider the game where the player I has m pure strategies and the player II has n pure strategies. A mixed strategy for the player I can be denoted by a column vector of probabilities $\mathbf{p} = (p_1, p_2, \ldots, p_m)^T$. Similarly, a mixed strategy for the player II is a probability vector $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$. The sets of mixed strategies of players I and II will be denoted respectively by X^* and Y^*, and given by:

$$X^* = \{\mathbf{p} = (p_1, p_2, \ldots, p_m)^T : p_i \geq 0,$$
$$\text{where } i = 1, \ldots, m \text{ and } \sum_{i=1}^{m} p_i = 1\},$$
$$Y^* = \{\mathbf{q} = (q_1, q_2, \ldots, q_n)^T : q_i \geq 0,$$
$$\text{where } i = 1, \ldots, n \text{ and } \sum_{i=1}^{m} q_i = 1\}.$$

It is worth to observe that the m-dimensional unit vectors $e_k \in X^*$, where the k-th element is one and the

others are zeros, can be identified as the pure strategies, the choice of the k-th row in the game matrix A. So, we can suppose that $X \subset \mathrm{X}^*$.
If $\mathbf{p} = (p_1, p_2, ..., p_m)^T$ is the mixed strategy adopted by player I and the j-th column is the choice of player II, then, on average, the *payoff* for player I is:

$$\sum_{i=1}^{m} p_i a_{ij}. \tag{1}$$

Similarly if player II uses $\mathbf{q} = (q_1, q_2, \ldots, q_n)^T$, and player I chooses the i-th row, then, on average, the *payoff* for player I is:

$$\sum_{j=1}^{n} q_j a_{ij}. \tag{2}$$

Generally, if player I uses the mixed strategy **p** and player II uses the mixed strategy **q**, on average, the *payoff* for player I is:

$$A(\mathrm{p}, \mathrm{q}) = \mathrm{p}^T A\mathrm{q} = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j. \tag{3}$$

Now, suppose that the player I have discovered, in advance, the mixed strategy $\mathbf{q} \in Y^*$ of the player II. In this case, player I can choose the i-th line that maximizes (2) or, equivalently, he can choose some $\mathbf{p} \in X^*$ that maximizes (3). This strategy is known as the **best answer (or the Bayes strategy) against q**:

$$\max_{1\leq i\leq m} \sum_{j=1}^{n} a_{ij} q_j = \underset{p\in X^*}{max}\, \mathrm{p}^T A\mathrm{q}. \tag{4}$$

To prove the equality (4) it is sufficient to note that the right side of equality is the maximum of $\mathbf{p}^T A\mathbf{q}$ among all $\mathbf{p} \in X^*$. Then, since $X \in X^*$, the left hand side must be less than or equal to the right hand side:

$$\max_{1\leq i\leq m} \sum_{j=1}^{n} a_{ij} q_j \leq \underset{p\in X^*}{max}\, \mathrm{p}^T A\mathrm{q}.$$

Moreover, as (3) is the average amount in (2), the maximum value of (2), precisely the left side of (4), must be greater than or equal to

$$\underset{p\in X^*}{max}\, \mathrm{p}^T A\mathrm{q} = \max_{1\leq i\leq m} \sum_{j=1}^{n} a_{ij} q_j \geq \underset{p\in X^*}{max}\, \mathrm{p}^T A\mathrm{q}.$$

Then, $\max_{1\leq i\leq m} \sum_{j=1}^{n} a_{ij} q_j = \underset{p\in X^*}{max}\, \mathrm{p}^T A\mathrm{q}$.
Now suppose that the new determination of the game is obligatory disclosure of the strategy of the player II to the player I. In that case, knowing the possibility of the player I to use Bayes strategies, player II may resort to a strategy called **minimax strategy**, to minimize his/her maximum average loss, regardless of the strategy adopted by the player I:

$$\bar{V} = \min_{q\in Y^*} \max_{1\leq i\leq m} \sum_{j=1}^{n} a_{ij} q_j \tag{5}$$

$$= \min_{\mathrm{q}\in Y^*} \max_{\mathrm{p}\in X^*} p^T A q$$

where $\bar{V}$ is called **superior value of the game** $(X$, Y, $A)$.
Regarding player II, the reasoning is similar to player I. Then its **best reply (or the Bayes strategy) against p** is defined by:

$$\min_{1\leq j\leq n} \sum_{i=1}^{m} p_i a_{ij} = \min_{q\in Y^*} p^T A q \tag{6}$$

and the minimax, strategy is given by

$$\underline{V} = \max_{p\in X^*} \min_{1\leq j\leq n} \sum_{i=1}^{m} p_i a_{ij} \tag{7}$$

$$= \max_{p\in X^*} \min_{q\in Y^*} p^T A q,$$

stands for the **lower value of the game**.
To prove the existence of the superior value for finite games, it is worth noting that (4), the maximum of m linear functions of **q**, is a continuous function of **q** and, since Y^* is a closed set, this function necessarily assumes its minimum over Y^* at some point of Y^* (Ferguson [4]). Regarding the lower value of the game (X, Y, A), the reasoning for its existence is similar to the superior value.
The proof for $\underline{V} \leq \bar{V}$ can be done by absurd since, assuming the hypothesis $\underline{V} > \bar{V}$, it means that player II can lose on average more than $\bar{V}$ or player I can earn on average less than $\underline{V}$. It is a contradiction. Finally, the minimax theorem states that, for finite games, $\underline{V} = \bar{V}$ (Ferguson [4]). In this case $V = \underline{V} = \bar{V}$ is called the **value of the game** and the mixed strategies used by the players that ensure their return are called **optimal strategies**. If V is zero we say that the game is fair. If V is positive, the game is favorable to the player I, and if V is negative, the game is favorable to the player II. Solving a game means finding its value and, at least, one optimal strategy for each player.

3 Strategic Models for the secretary problem variations

First variation. Since the opponent can choose any row at random from the cyclical Latin square $n \times n$, then for n candidates to be ordered by the opponent, the cyclical Latin square is presented as follows:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ n-1 & 1 & 2 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 3 & 4 & \cdots & n & 1 \end{pmatrix}$$

Since player I succeeds when he hires the best secretary, then to construct the *payoff* matrix, one sets the value 1 when the observer hires the best secretary and 0 when he hires any other but the best. The set of pure strategies for the observer is given by $X = 1, 2, \dots, n$, where 1 means hiring the first interviewee, 2 means hiring the second interviewee, and so on. In turn, the set of pure strategies for the opponent is $Y = L_1, L_2, ..., L_n$, where L_1 means to choose the first row of the Latin square, L_2 means to choose the second row, and so on. The matrix of the game is:

$$\begin{array}{c c} \begin{array}{ccccc} L_1 & L_2 & L_3 & & L_n \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \end{array}$$

Second variation. Since the opponent has the power to choose only the position of the best candidate, then in the modeling process of this variation of the problem, the strategy of placing the best candidate on the r-th position is denoted by T_r, and T is the mixed strategy that chooses T_r with probability p_r. Meanwhile the strategy of ignoring the first i candidates and then choose the first candidate better than the previous, is denoted by S_i, and S is the mixed strategy that chooses S_i with probability p_i. Therefore, if the observer uses the strategy S_i and the opponent uses the strategy T_r, the probability of the observer to win is 0 if $i \geq r$ (i.e., if the best candidate is between the first i candidates ignored by the observer) and $i/(r-1)$ if $i < r$.
Summarizing, whenever the players use their pure strategies, that is, the observer uses the strategy S_i and the opponent uses the strategy T_r, the average *payoff* is $i/(r-1)$ when $i < r$. Observer will succeed with his strategy S_i if the best candidate among the first $r-1$ candidates is among the first i candidates.
To demonstrate this fact, consider a set with $r-1$ distinct balls among which there is one considered to the best. If i balls from that set were placed in a box, the total number of possible ways is the i-combination of $r-1$ balls: $C_i^{r-1} = \begin{pmatrix} r-1 \\ i \end{pmatrix}$.
Now, one wants the best ball to be among the i balls placed in the box. The total number of possible ways of putting $i-1$ balls plus the best ball from $r-1$ distinct balls in a box is the $(i-1)$-combination of $r-2$ balls: $C_i^{r-1} = \begin{pmatrix} r-2 \\ i-1 \end{pmatrix}$.
Thus, the probability of success, that is, the probability of the best ball to be among the i balls in the box is $P_{[sucess]} = \dfrac{C_{i-1}^{r-2}}{C_i^{r-1}} = \dfrac{i}{r-1}$. So, when $i < r$ the probability of the observer to hire the best secretary is $i/(r-1)$. Therefore, the game matrix for the observer is:

	T_1	T_2	T_3	$\dots$	T_{n-1}	T_n
S_1	0	1	1/2	$\dots$	1/(n-2)	1/(n-1)
S_2	0	0	1	$\dots$	2/(n-2)	2/(n-1)
S_3	0	0	0	$\dots$	3/(n-2)	3/(n-1)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
S_{n-1}	0	0	0	$\dots$	0	1

Third variation. Figure 1 illustrates the situation in which the opponent may present, with probability $1/2$, the best (lowest post) or the worst (highest post) secretary, among those who have not yet been interviewed. One can note that the numbers represent the ranks of the candidates and the ways on the graph represent all possible presentation sequences of the candidates.
Regarding strategies, the observer has n pure strategies (hiring the first secretary, the second, ..., the n-th) and the opponent has 2^{n-1} pure strategies (the number of paths of the graph).
Thus, the game matrix can be represented as follows: each entry is the rank of the hired secretary (i.e., 1, if the best secretary is hired, 2, if the second best secretary is hired, and so on):

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 & n & n & \dots & n \\ 2 & \dots & 2 & n & n & 1 & \dots & 1 & \dots & n-1 \\ \vdots & & & & & \ddots & & & & \vdots \\ n-1 & n & \dots & & & \dots & & & & 2 \\ n & n-1 & \dots & & & \dots & & & & 1 \end{pmatrix}$$

The first row of the matrix indicates that the observer hires the first interviewee, so he hires the best candidate (in the matrix represented by the number 1) with probability $1/2$ or hires the worst candidate, n-th, also with probability $1/2$. In the second row, the observer hires the second interviewee, that is, he hires the best

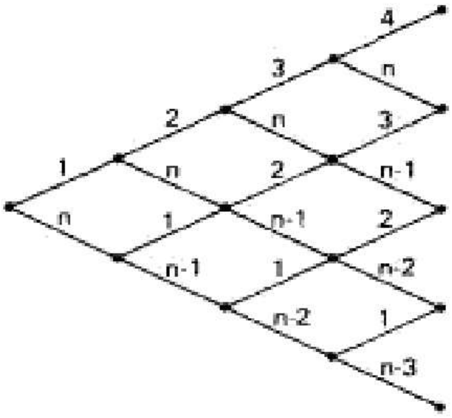

Figure 1: Graphical sequences representing all possible using the strategy of the opponent.

candidate (1), or the second better (2), or the worst (n), or even the second worst ($n-1$), with probability $1/4$ each. Generally, on the k-th row, the observer hires the k-th interviewee, that is, he hires the best (1), or the second better (2), ..., or the k-th better, or the worst (n), or the second worst ($n-1$), ... or even the k-th worst ($n-(k-1)$), with probability $1/2^k$ each. The matrix columns present all paths in the graph.

4 Solving the three variations of the secretary problem via Linear Programming

Considering the variations of the secretary problem according to the player's I perspective is determining p_1, ..., p_m in order to maximize (6) subject to the restriction that $\mathbf{p} \in X^*$, which formally can be written as the following optimization model:

$$\begin{aligned} &\max \min_{1\leq j\leq n} \sum_{i=1}^{m} p_i a_{ij} \\ &\text{s.t.} \sum_{i=1}^{m} p_i = 1 \\ &p_i \geq 0,\ i = 1,\ldots,m. \end{aligned}$$

Although the restrictions are linear, the objective function is not linear due to the minimization operator. However, this can be rounded by maximizing an auxiliar variable v, which in turn must be less than the objective function, i.e., $v \leq \min_{1\leq j\leq n} \sum_{i=1}^{m} p_i a_{ij}$. Thus, we have the following linear programming (LP) model

$$\begin{aligned} &\max v \\ &\text{s.t.} \sum_{i=1}^{m} p_i a_{ij} \geq v,\ j = 1,\ldots,8 \\ &\sum_{i=1}^{m} p_i = 1 \\ &p_i \geq 0,\ i = 1,\ldots,m, \end{aligned}$$

This model is able not only to return the value of game but also some mixed optimal strategy for player I. One way to solve a problem of linear programming is via the simplex method, implemented in this work through the programming package in Java (JDK $1.5.0$ version 8) and the interface GLPK 4.8 NYI, whose partial result was as follows (Table 1):

Table 1: Values of objective function in the simplex outputs that represent the values of the game for three variations of the secretary problem, where $n = 2,\ldots,5$ is the number of strategies for player II (opponent).

	Amount of strategies for player II (opponent)			
Var	2	3	4	5
1	OPTIMAL Objective: 0.5 (MAXimum)	OPTIMAL Objective: 0.333 (MAXimum)	OPTIMAL Objective: 0.25 (MAXimum)	OPTIMAL Objective: 0.2 (MAXimum)
2	OPTIMAL Objective: 2 (MAXimum)	OPTIMAL Objective: 2.5 (MAXimum)	OPTIMAL Objective: 2.833 (MAXimum)	OPTIMAL Objective: 3.083 (MAXimum)
3	OPTIMAL Objective: 1.5 (MAXimum)	OPTIMAL Objective: 2 (MAXimum)	OPTIMAL Objective: 2.5 (MAXimum)	OPTIMAL Objective: 3 (MAXimum)

5 Theoretical validation via game theory

First variation. Let $p^T = \begin{pmatrix} p_1 & p_2 & \ldots & p_n \end{pmatrix}$ be a mixed strategy for the observer. Then, if opponent uses an uniform mixed strategy $q^T = (1/n\, 1/n\, \ldots\, 1/n)$. That is, his intention is to choose a line at random from the $n \times n$ cyclical Latin square. The mean *payoff* is then:

$$\mathrm{p}^T A \mathrm{q} = \begin{pmatrix} p_1 & p_2 & \ldots & p_n \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{pmatrix} = \frac{1}{n}\left(p_1 + p_2 + \ldots + p_n\right) = \frac{1}{n}$$

The above result shows that, independent on the strategy adopted by the observer, if the opponent chooses a line at random from the $n \times n$ cyclical Latin square he reduces the probability of success to the minimum $1/n$.

Second variation. The probability of the observer to hire the best candidate using the strategy S_i and the opponent using the mixed strategy $T = (p_1, p_2, \ldots, p_n)$ is, therefore, $\sum_{r=i+1}^{n} p_r \dfrac{i}{r-1}$.

The opponent wants, for sure, to choose a mixed strategy T that ensures minimizing the probability of success of the observer. For achieving such goal it is necessary to use a procedure that equals the mean gains of both players. Then, Player II wants to choose an strategy T, that is, to determine the values of p_i such that his gain is the same if the Player I chooses line i or line $i+1$, that is:

$$\sum_{j=i+1}^{n} p_j \frac{i}{j-1} = \sum_{j=i+2}^{n} p_j \frac{i+1}{j-1},$$

which can be solved by recurrence, as follows: For $i = n-2$, gives:

$$\begin{aligned} &\sum_{j=n-1}^{n} p_j \frac{n-2}{j-1} = \sum_{j=n}^{n} p_j \frac{n-1}{j-1}; \\ &p_{n-1} + p_n \frac{n-2}{n-1} = p_n; \\ &p_{n-1} = p_n \left(1 - \frac{n-2}{n-1}\right); \\ &p_{n-1} = p_n \frac{1}{n-1}. \end{aligned}$$

For $i = n-3$, gives:

$$\begin{aligned} &\sum_{j=n-2}^{n} p_j \frac{n-3}{j-1} = \sum_{j=n-1}^{n} p_j \frac{n-2}{j-1}; \\ &p_{n-2} + p_{n-1} \frac{n-3}{n-2} + pn \frac{n-3}{n-1} \\ &= p_{n-1} + p_n \frac{n-2}{n-1}; \\ &p_{n-2} = p_n \left(\frac{1}{(n-1)(n-2)} + \frac{1}{n-1}\right); \\ &p_{n-2} = p_n \frac{1}{n-2}, \end{aligned}$$

and so on, what yields following generalization: $p_j = p_n \frac{1}{j}$.

Since $\sum_{j=1}^{n} p_j = 1$, follows that

$$\begin{aligned} &p_1 + p_2 + \ldots + p_{n-1} + p_n = 1; \\ &p_n + p_n \frac{1}{2} + \ldots + p_n \frac{1}{n-1} + p_n = 1; \\ &\left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + 1\right) p_n = 1; \\ &p_n = \left(1 + \sum_{i=2}^{n-1} \frac{1}{i}\right)^{-1}. \end{aligned}$$

Setting $p_j = K/j$ and $p_n = K$, where

$$K = \left(1 + \sum_{i=1}^{n-1} \tfrac{1}{i}\right)^{-1}.$$

In this case the probability of success is K. Similarly, if the opponent uses the strategy T_r and the observer uses a mixed strategy $S = (\pi_1, \pi_2, \ldots, \pi_{n-1})$, the probability of the observer to hire the best candidate is
$\sum_{i=1}^{r-1} \pi_i \frac{i}{r-1}$.

The observer wants, of course, choosing a mixed strategy S that ensures maximizing his probability of success. Using the same reasoning above, $\pi_i = K/i$ for $i = 1, 2, \ldots, n-1$. The mean return for the observer using this strategy is then K. This is therefore the real minimax solution of this game.

Third variation. The idea is to count the occurrences of distinct ranks in each row of the game matrix. Since the first row corresponds to hiring in the first interview, the worst or the best candidate will be chosen with probability $1/2$. Thus, the worst candidate (n) is presented C_0^1 times and the best candidate (1) is presented C_1^1 times. As the opponent has 2^{n-1} pure strategies or 2^{n-1} ways of sorting the candidates to be submitted, the ranks 1 and n appear in the first row $(2^{n-2} C_0^1)$ times each.

In general, considering the k-th row of the matrix, the worst candidate (n) and the k-th best candidate (k) are presented C_0^{k-1} times, the second worst $(n-1)$ and $(k-1)$-th best $(k-1)$ are presented C_1^{k-1} times, ..., the $(k-1)$-th worst $(n-(k-2))$ and the second best (2) are presented C_{k-2}^{k-1} times, and, finally, (k)-th worst $(n-(k-1))$ and the best (1) are presented C_{k-1}^{k-1} times (according to the paths on the Figure 1). Thus, the ranks $1, 2, \ldots, k, (n-(k-1)), \ldots, (n-1), n$, appear in the k-th row of the matrix

$$\begin{aligned} &1/2^k \cdot 2^{n-1} \cdot C_{k-1}^{k-1}, 1/2^k \cdot 2^{n-1} \cdot C_{k-2}^{k-1}, \ldots, \\ &1/2^k \cdot 2^{n-1} \cdot C_0^{k-1}, 1/2^k \cdot 2^{n-1} \cdot C_{k-1}^{k-1}, \ldots, \\ &1/2^k \cdot 2^{n-1} \cdot C_1^{k-1} \text{ e } 1/2^k \cdot 2^{n-1} \cdot C_0^{k-1} \end{aligned}$$

times, respectively. Another point that should be stressed is that the sum of the elements in each row of the matrix is constant. Considering the generic form presented above, one can confirm that. Adding up the elements of the k-th row of the matrix, gives:

$$\begin{aligned} &1\left[2^{n-(k+1)} C_{k-1}^{k-1}\right] + 2\left[2^{n-(k+1)} C_{k-2}^{k-1}\right] + \ldots + \\ &(k-1)\left[2^{n-(k+1)} C_1^{k-1}\right] + k\left[2^{n-(k+1)} C_0^{k-1}\right] \\ &+(n-(k-1))\left[2^{n-(k+1)} C_{k-1}^{k-1}\right] + (n-(k-2)) \\ &\left[2^{n-(k+1)} C_{k-2}^{k-1}\right] + \ldots + \\ &(n-1)\left[2^{n-(k+1)} C_1^{k-1}\right] + n\left[2^{n-(k+1)} C_0^{k-1}\right] \end{aligned}$$

Factoring up this term, gives:

$$2^{n-(k+1)}[n \sum_{i=0}^{k-1} C_i^{k-1} + k C_0^{k-1} + (k-2) C_1^{k-1} +$$

$$+ \ldots + (-k+4) C_{k-2}^{k-1} + (-k+2) C_{k-1}^{k-1}] \qquad (8)$$

But, $\sum_{i=0}^{k-1} C_i^{k-1} = 2^{k-1}$; $C_0^{k-1} = C_{k-1}^{k-1}$; $C_1^{k-1} = C_{k-2}^{k-1}$

and so on. Therefore, the following grouping can be set:

$$\begin{aligned} &kC_0^{k-1} + (-k+2)C_{k-1}^{k-1} = \\ &= 2C_0^{k-1} = C_0^{k-1} + C_{k-1}^{k-1}; \end{aligned}$$

$$\begin{aligned} &(k-2)C_1^{k-1} + (-k+4)C_{k-2}^{k-1} = \\ &= 2C_1^{k-1} = C_1^{k-1} + C_{k-2}^{k-1} \end{aligned}$$

and so on, giving:

$$\begin{aligned} &C_0^{k-1} + C_1^{k-1} + \ldots + C_{k-2}^{k-1} + C_{k-1}^{k-1} = \\ &= \sum_{i=0}^{k-1} C_i^{k-1} = 2^{k-1} \end{aligned}$$

Returning to (8):

$$2^{n-(k+1)}[n \underbrace{\sum_{i=0}^{k-1} C_i^{k-1}}_{2^{k-1}} +$$

$$+ \underbrace{C_0^{k-1} + C_1^{k-1} + \ldots + C_{k-2}^{k-1} + C_{k-1}^{k-1}}_{2^{k-1}}]$$

$$= 2^{n-(k+1)}[n(2^{k-1}) + 2^{k-1}] = 2^{n-2}(n+1).$$

Thus, any row in the matrix adds to $2^{n-2}(n+1)$.
Next it will be proved that such strategy, used by the opponent, inhibits the observer to take any strategy that brings him some advantage, that is, no matter what strategy the observer uses, he always will get a mean post $(n+1)/2$. If the observer uses a mixed strategy

$$p^T = \begin{pmatrix} \pi_1, & \pi_2, & \ldots, & \pi_n \end{pmatrix}$$

and the opponent uses the uniform mixed strategy:

$$q^T = \begin{pmatrix} 1/2^{n-1}, & 1/2^{n-1}, & \ldots, & 1/2^{n-1} \end{pmatrix}$$

, the mean *payoff* for the observer is given by:

$$\mathrm{p}^T A\mathrm{q} = (\pi_1,\ \pi_2,\ \ldots,\ \pi_n)\left(\tfrac{1}{2^{n-1}}\right) \begin{pmatrix} 2^{\text{n-2}}(\text{n+1}) \\ 2^{\text{n-2}}(\text{n+1}) \\ \vdots \\ 2^{\text{n-2}}(\text{n+1}) \end{pmatrix}$$

$$= \left(\tfrac{1}{2^{n-1}}\right) 2^{\text{n-2}}(\text{n+1})[\underbrace{\pi_1 + \pi_2 + \ldots + \pi_n}_{1}] = \tfrac{n+1}{2}$$

from where the result follows.
It is worth noting that this result is demonstrated by Chow et al. (1964) using the sophisticated theory of Martingales. Table 2 compares the these results with those in previous section showing the optimality of the strategies evaluated.

Table 2: Comparison of the game theory and linear programming approach of the three variations of the secretary problem.

	Secretary problem approach	
Var	Game Theory	L.P.
1	$\frac{1}{n}$	$V = \frac{1}{n}$
2	$K = \left(1 + \sum_{i=1}^{n-1} \frac{1}{i}\right)^{-1}$	$V = \left(1 + \sum_{i=1}^{n-1} \frac{1}{i}\right)^{-1}$
3	$\frac{(n+1)}{2}$	$V = \frac{(n+1)}{2}$

6 Conclusions

The basic tools of Game Theory is enough to solve the three variations of the secretary problem presented here. This approach has the advantage of being elementary and didactic, avoiding the heavy formalism of the probabilistic approach as the Martingale theory.
Moreover, it represents them by them strategic way and via linear programming models, enabling not only to validate the theoretical analysis of specific strategies adopted by the players but also to assess how fast and efficient the optimality of several more complex strategies are.

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